Sree Kerala Varma College, Thrissur

Functional Analysis I

Assignment 3

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1. Let X_1 be a closed subspace and X_2 be a finite dimensional subspace of a normed space X. Show that the sum $X_1 + X_2 = \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$ is closed in X. (Hint: Proof of the theorem: "Every finite dimensional subspace of a normed space is complete"). Show by a counterexample that the sum of two closed sets need not be closed.

2. Let $X = C([a,b])$. For $x \in X$, let $\|x\|_1 = \int_a^b |x(t)|\,dt$ and $\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$. Show that the norm $\|.\|_\infty$ is stronger than but not equivalent to the norm $\|.\|_1$.

3. Let $X = C^1([a,b])$, the linear space of all continuously differentiable functions on $[a,b]$. Denote the derivative of x by x'. For $x \in X$, let

$$\|x\| = \|x\|_\infty + \|x'\|_\infty \quad \text{and} \quad \|x\|_0 = |x(a)| + \|x'\|_\infty.$$

Show that $\|.\|$ and $\|.\|_0$ are equivalent norms on X by showing that $\|x\|_0 \leq \|x\| \leq (b-a+1)\|x\|_0$ for all $x \in X$.

5. Let X be the linear space of all polynomials in one variable with coefficients from $\mathbb{K}$. For $p \in X$ with $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, let

$$\|p\| = \sup\{|p(t)| : 0 \leq t \leq 1\},$$

$$\|p\|_1 = |a_0| + \cdots + |a_n|,$$

$$\|p\|_\infty = \max\{|a_0|, \cdots, |a_n|\}.$$

Show that $\|p\| \leq \|p\|_1$ and $\|p\|_\infty \leq \|p\|_1$ for all $p \in X$. Also show that $\|.\|$ and $\|.\|_1$ are not equivalent and $\|.\|$ and $\|.\|_\infty$ are not comparable.

6. Show that the closed unit ball of ℓ^p; $1 \leq p \leq \infty$ is not compact by producing a sequence in the ball having no convergent subsequence.

7. Let c_{00} denote the linear space of all $\mathbb{K}$-valued sequences $x = (x(1), x(2), \cdots)$ with $x(j) = 0$ for all but finitely many j. For $x \in c_{00}$ and for $1 \leq p \leq \infty$, define the norm

$$\|x\|_p = \begin{cases} \left(\sum_{j=1}^\infty |x(j)|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup_j |x(j)| & \text{if } p = \infty \end{cases}$$

Show that $(c_{00}, \|.\|_p)$ is a subspace of ℓ^p, but not closed in ℓ^p. Also show that for $1 \leq p < r \leq \infty$, the norm $\|.\|_p$ is stronger than but not equivalent to the norm $\|.\|_r$.